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REDUCTION TO CANONICAL FORMS AND THE STOKES PHENOMENON IN THE THEORY OF LINEAR DIFFERENCE EQUATIONS*

G. K. IMMINK†

Abstract. Previous results concerning the existence of right inverses of linear difference operators on Banach Spaces of holomorphic functions are extended. The Stokes phenomenon is analyzed for a class of very singular linear difference equations.

Key words. linear difference operator, right inverse, asymptotic expansion, canonical form, Stokes phenomenon, connection matrix

AMS(MOS) subject classification. 39A10

Introduction. This paper is concerned with homogeneous linear difference equations of the type

$$(0.1) \quad y(s+1) - A(s)y(s) = 0,$$

where s is a complex variable and A is an $n \times n$ matrix function, meromorphic at ∞ . More precisely, we shall assume that $A \in \text{Gl}(n; \mathbb{C}\{s^{-1}\}[s])$, $n \in \mathbb{N}$. We are interested in the global asymptotic properties of solutions of (0.1), i.e., their behaviour as $s \rightarrow \infty$ in an arbitrary direction.

It should be noted here that, in general, solutions of (0.1) are not analytic in a (reduced) neighbourhood of ∞ . However, it is easily seen from (0.1) that any solution, analytic in a left half plane, can be continued analytically to a region of the form $-U(R)$, where

$$(0.2) \quad U(R) = \{s \in \mathbb{C} : |s+x| \geq R \forall x \geq 0\}, \quad R > 0.$$

Similarly, the relation

$$y(s) = A(s)^{-1}y(s+1)$$

implied by (0.1), shows that any solution, analytic in a right halfplane, can be continued analytically to a region of the form $U(R)$, $R > 0$. Therefore, we shall consider the asymptotic behaviour of solutions of (0.1) in regions of either type.

The usual approach to this kind of problem is the following. First, the existence of fundamental solutions of the equation with a prescribed asymptotic behaviour, in different sectors covering a neighbourhood of ∞ is established. Next, the relations between these solutions are studied.

For example, let Y_1 and Y_2 be holomorphic fundamental solutions of (0.1), admitting the same asymptotic representation in a left and an upper halfplane, respectively. The connection matrix P is defined by

$$(0.3) \quad Y_1 = Y_2 P.$$

As $Y_1(s+1)Y_1(s)^{-1} = Y_2(s+1)Y_2(s)^{-1} = A(s)$, P is a periodic function of period 1. Obviously, (0.3) defines the analytic continuation of Y_1 to an upper halfplane and knowledge of P implies knowledge of the asymptotic behaviour of Y_1 in this upper half plane.

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The asymptotic representations that play a role in the study of (0.1) are the so-called formal solutions of this equation. It is known that (0.1) possesses a formal fundamental solution of the form

$$\hat{Y}(s) = \hat{F}(s) \operatorname{diag} \{s^{G_1} \exp q_1(s), \dots, s^{G_m} \exp q_m(s)\},$$

where $\hat{F} \in \operatorname{Gl}(n; \mathbb{C}[[s^{-1/p}]][[s^{1/p}]])$ for some $p \in \mathbb{N}$, $m \in \mathbb{N}$ ($m \leq n$), G_j is a constant matrix and

$$(0.4) \quad q_j(s) = d_j s \log s + \sum_{h=1}^p \mu_{j,h} s^{h/p}$$

with $d_j \in \mathbb{Q}$ and $\mu_{j,h} \in \mathbb{C}$ for each $j \in \{1, \dots, m\}$, $h \in \{1, \dots, p\}$ (cf. [2], [13], [16]). All constants figuring in this representation are uniquely determined by the matrix function A , except for $\mu_{j,p}$, which is determined up to a multiple of $2\pi i$ (this is related to the fact that any matrix solution of (0.1), when multiplied from the right with a periodic matrix function, remains a solution of (0.1)).

If $d_i = d_j$ for all $i, j \in \{1, \dots, m\}$, the numbers

$$\frac{1}{p} \operatorname{degr} (q_i - q_j)$$

will be called the “levels” of the difference equation. The most difficult case to deal with is when $d_i \neq d_j$ for at least one pair (i, j) with $i \neq j$. In this case we will say that the “level 1^+ ” is present in (0.1). If $d_i = d_j$ for all $i \neq j$, we shall, with a slight abuse of terminology, speak of a difference equation of level 1^+ (cf. [4]).

Existence theorems for solutions of homogeneous linear difference equations with a prescribed asymptotic behaviour have been the subject of various studies since the beginning of this century (cf. [1], [3], [5], [7], [12]). The most general result so far is a theorem by Birkhoff and Trjitzinsky (cf. [1]). It states that, in every quadrant Γ of the form

$$\Gamma = \left\{ s \in \mathbb{C} : k \frac{\pi}{2} \leq \arg (s - s_0) \leq (k+1) \frac{\pi}{2}; |s| \geq R \right\}, \quad k \in \mathbb{Z}, \quad s_0 \in \mathbb{C}, \quad R > 0,$$

there exists a holomorphic fundamental solution of (0.1), represented asymptotically by a given formal fundamental solution as $s \rightarrow \infty$ in Γ , provided R is sufficiently large. However, the proof of this result contains some inaccuracies and its correctness has been questioned.

In [7] we have derived existence theorems for both linear and nonlinear difference equations using a method developed by Hukuhara, Sibuya, Malgrange and others, for analogous problems in the theory of differential equations (cf. [6], [11], [14], [17]). It is based on the existence of right inverses of linear difference operators on Banach spaces of functions that are holomorphic in suitable (“proper”) regions of the complex plane. Later we realized that the class of “proper” regions considered in [7], at least in the presence of level 1^+ , was too restricted, and that our results could be easily extended. This is explained in § 2 of the present paper. These generalizations permit us to give a straightforward proof of the theorem of Birkhoff and Trjitzinsky, thereby settling the first half of our problem.

Next, we turn to the second part: the connection problem or Stokes phenomenon, i.e., the change in asymptotic behaviour of solutions of the equation, as they are continued analytically beyond certain “maximal regions.” This phenomenon has been studied in [8], under “generic” conditions, and in particular the link with the theory of resurgent functions was explained. It is closely related to the problem of analytic

classification of difference equations, which has been solved by Ecalle in [4]. Nevertheless, the precise nature of the Stokes phenomenon in the most general case of (0.1) has remained somewhat mysterious. The results presented in § 2 also contribute to a better understanding of this phenomenon.

The maximal regions in which a solution of (0.1) may be represented asymptotically by a given formal solution are bounded by curves of the form

$$\operatorname{Re} \{q_i(s) - q_j(s)\} = c, \quad i, j \in \{1, \dots, m\}, \quad i \neq j.$$

We shall call these curves Stokes curves of level k if $d_i = d_j$ and $1/p \operatorname{degr}(q_i - q_j) = k$, and Stokes curves of level 1^+ if $d_i \neq d_j$. Due to the infinite number of possible determinations of the $\mu_{i,p}$ in (0.4), there is a countably infinite number of Stokes curves of the levels 1 and 1^+ (we do not distinguish between curves that differ only in the value of c). Whereas two Stokes curves of a level less than or equal to 1 generally have distinct limiting directions, those of level 1^+ all have the same limiting directions, viz. those of the positive and negative imaginary axis. This makes the analysis of the Stokes phenomenon more delicate in the presence of the level 1^+ than otherwise. In order to distinguish between different solutions, we must take into consideration their behaviour along curves of the type

$$\operatorname{Re} s(\log s + i\theta) = c, \quad \theta \in \mathbb{R}.$$

Section 4 deals with the Stokes phenomenon of homogeneous linear difference equations of level 1^+ . We study the properties of the periodic matrix functions connecting two fundamental solutions of (0.1) represented asymptotically by the formal fundamental solution in different maximal regions (a method to compute the leading parts of these connection matrices from the asymptotic behaviour of the coefficients of the formal fundamental solution is discussed in [9]).

Throughout this paper, we have restricted ourselves to “classical” asymptotic expansions, just as in [7], however, all statements remain valid (with slight modifications) when these expansions are replaced by asymptotic expansions with suitable Gevrey-type error bounds (cf. also [10]).

1. Definitions and notation.

1.1. Classes of holomorphic functions admitting an asymptotic power series representation. In order to describe the asymptotic properties of solutions of (0.1), we introduce families of closed unbounded regions of \mathbb{C} , indexed by a parameter $R \in \mathbb{R}^+$, which measures the distance to the origin. We define classes of holomorphic functions on these regions, admitting an asymptotic expansion with uniform error bounds.

We shall restrict our attention to subregions of $\mathbb{C} \setminus \mathbb{R}^-$. All results obtained for this type of regions can be easily “translated” into analogous statements for corresponding regions in $\mathbb{C} \setminus \mathbb{R}^+$, by means of the following equality:

$$y(s+1) - A(s)y(s) = -A(s)\{\tilde{y}(-s) - \tilde{A}(-s-1)\tilde{y}(-s-1)\},$$

where \tilde{A} and \tilde{y} are defined by

$$\tilde{A}(s) = A(-s-1)^{-1}, \quad \tilde{y}(s) = y(-s).$$

DEFINITION 1.1.1. An asymptotic set S of closed regions is a decreasing set of closed unbounded regions $S(R)$ of the complex plane, defined for all $R > 1$, with the property that $d(S(R), 0) \rightarrow \infty$ as $R \rightarrow \infty$.

DEFINITION 1.1.2. Let I be an index set and let $S_i = \{S_i(R), R > 1\}$ be an asymptotic set of closed regions for all $i \in I$. By $\bigcup_{i \in I} S_i$ and $\bigcap_{i \in I} S_i$ we shall denote

the sets

$$\left\{ \bigcup_{i \in I} S_i(R), R > 1 \right\}, \quad \left\{ \bigcap_{i \in I} S_i(R), R > 1 \right\},$$

respectively. A set of the first type will be called an asymptotic set.

If I is a finite set then $\bigcup_{i \in I} S_i$ is again an asymptotic set of closed regions. If, in addition, $\bigcap_{i \in I} S_i(R) \neq \emptyset$ for all $R > 0$, then also $\bigcap_{i \in I} S_i$ is an asymptotic set of closed regions.

DEFINITION 1.1.3. Let S be an asymptotic set of closed regions. By $\mathcal{M}(S)$ we shall denote the set of all functions f on \mathbb{C} with the following properties:

(i) There exists a positive number R such that f is continuous on $S(R)$ and holomorphic in $\text{int } S(R)$.

(ii) There exist an integer h_0 , a positive integer p and complex numbers a_h , $h \in \mathbb{Z}$, such that, for all $N \in \mathbb{N}$,

$$\sup_{s \in S(R)} \left| s^{N/p} \left\{ f(s) - \sum_{h=h_0}^{N-1} a_h s^{-h/p} \right\} \right| < \infty.$$

If these conditions are fulfilled we write

$$\sum_{h=h_0}^{\infty} a_h s^{-h/p} = \hat{f} \quad \text{and} \quad f \sim \hat{f}, \quad s \rightarrow \infty, \quad s \in S(R).$$

Remark. Different functions that coincide on $S(R)$ for some $R > 0$, will be identified. More precisely, a “function” f is to be thought of as a representative of the equivalence class of all functions g with the property that there exists a positive number R such that $g(s) = f(s)$ for all $s \in S(R)$.

Let I be a finite index set and suppose that, for every $i \in I$, S_i is an asymptotic set of closed regions. Then, obviously, $\mathcal{M}(\bigcup_{i \in I} S_i)$ is contained in $\bigcap_{i \in I} \mathcal{M}(S_i)$. If, in addition, $\text{int } \bigcup_{i \in I} S_i(R) = \bigcup_{i \in I} \text{int } S_i(R)$ for all $R > 1$, then it immediately follows that $\mathcal{M}(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{M}(S_i)$.

DEFINITION 1.1.4. Let I be an index set and, for every $i \in I$, let S_i be an asymptotic set of closed regions such that

$$\text{int } \bigcup_{i \in I} S_i(R) = \bigcup_{i \in I} \text{int } S_i(R) \quad \text{for all } R > 1.$$

Let $S = \bigcup_{i \in I} S_i$. By $\mathcal{M}(S)$ we shall denote the set

$$\mathcal{M}(S) = \bigcap_{i \in I} \mathcal{M}(S_i),$$

by $\mathcal{A}(S)$ the set of all $f \in \mathcal{M}(S)$ such that

$$\hat{f} \in \bigcup_{p \in \mathbb{N}} \mathbb{C}[[s^{-1/p}]],$$

and by $\mathcal{A}_0(S)$ the set of $f \in \mathcal{M}(S)$ with the property that $\hat{f} = 0$.

In the following definition some particular examples of asymptotic sets are given.

DEFINITION 1.1.5. For all $\alpha_0, \beta_0 \in [-\pi, \pi]$ such that $\alpha_0 < \beta_0$, we define the following asymptotic sets:

(i) The “closed sector” $S[\alpha_0, \beta_0]$ defined by

$$S[\alpha_0, \beta_0](R) = \{s \in \mathbb{C} \setminus \mathbb{R}^- : \alpha_0 \leq \arg(s + \text{Re}^{i\alpha}) \leq \beta_0 \text{ for all } \alpha \in [-\pi, \pi]\};$$

(ii) The “half-open sectors” $S[\alpha_0, \beta_0) = \bigcup_{\alpha_0 < \beta < \beta_0} S[\alpha_0, \beta]$ and $S(\alpha_0, \beta_0] = \bigcup_{\alpha_0 < \alpha < \beta_0} S[\alpha, \beta_0]$;

(iii) The “open sector” $S(\alpha_0, \beta_0) = \bigcup_{\alpha_0 < \alpha < \beta < \beta_0} S[\alpha, \beta]$.

If S is any of the sectors defined in (ii) and (iii), then \bar{S} will denote the closed sector $S[\alpha_0, \beta_0]$.

1.2. Canonical forms and formal invariants. We use the following notation:

$$K = \bigcup_{p \in \mathbb{N}} \mathbb{C}[s^{-1/p}][s^{1/p}], \quad \hat{K} = \bigcup_{p \in \mathbb{N}} \mathbb{C}[[s^{-1/p}]] [s^{1/p}].$$

Let S be an asymptotic set. If $A \in \text{End}(n; \mathcal{M}(S))$ and $F \in \text{Gl}(n; \mathcal{M}(S))$, or $A \in \text{End}(n; \hat{K})$ and $F \in \text{Gl}(n; \hat{K})$ we shall denote by A^F the matrix function

$$A^F(s) = F(s+1)^{-1} A(s) F(s).$$

By Δ_A we shall denote the linear difference operator defined by

$$\Delta_A y(s) = y(s+1) - A(s)y(s),$$

where y belongs to a suitable space of n -dimensional vector functions.

Let $A, B \in \text{Gl}(n; \mathcal{M}(S))$. The difference operators Δ_A and Δ_B are said to be formally equivalent if there exists a matrix function $F \in \text{Gl}(n; \hat{K})$ with the property that

$$\hat{A}^F = \hat{B}.$$

If $A \in \text{Gl}(n; \mathcal{M}(S))$ the difference operator Δ_A is known to be formally equivalent to a difference operator $\Delta_{\hat{A}}$ of the following particular type. The matrix function \hat{A} is block diagonal,

$$\hat{A} = \text{diag} \{ \hat{A}_1, \dots, \hat{A}_m \}, \quad m \in \mathbb{N}$$

with diagonal blocks of the form

$$\hat{A}_j(s) = \exp \{ q_j(s+1) - q_j(s) \} (1 + 1/s)^{G_j},$$

where, for $j \in \{1, \dots, m\}$,

$$(1.2.1) \quad q_j(s) = d_j s \log s + \sum_{h=1}^p \mu_{j,h} s^{h/p}, \quad p \in \mathbb{N}, \quad d_j \in \mathbb{Z}/p, \quad \mu_{j,1}, \dots, \mu_{j,p} \in \mathbb{C},$$

$0 \leq \text{Im } \mu_{j,p} < 2\pi$, and

$G_j = \gamma_j I_{n_j} + N_j$, $\gamma_j \in \mathbb{C}$, $0 \leq \text{Re } \gamma_j < 1/p$, $n_j \in \mathbb{N}$, N_j is a nilpotent

$n_j \times n_j$ matrix.

The matrix function \hat{A} is uniquely determined by A up to permutations of the diagonal blocks. We shall assume that the blocks are arranged in such a way that

$$d_i \leq d_j \quad \text{if } i < j, \quad i, j \in \{1, \dots, m\}.$$

DEFINITION 1.2.2. A matrix function \hat{A} with the above-mentioned properties will be called a canonical matrix or a canonical form of A .

The numbers d_j , γ_j , and $\mu_{j,h}$ ($j \in \{1, \dots, m\}$, $h \in \{1, \dots, p\}$) are formal invariants of the difference equation $\Delta_A y = 0$. They are uniquely determined by the matrix function A .

We shall further use the following notation:

$$Q = \text{diag} \{ q_1 I_{n_1}, \dots, q_m I_{n_m} \},$$

$$\hat{Y}(s) = e^{Q(s)} s^G \quad (\text{note that } \hat{A}(s) = \hat{Y}(s+1) \hat{Y}(s)^{-1}),$$

$$d(A) = \{d_1, \dots, d_m\},$$

$$k(A) = \{ \text{degr } q_j : j \in \{1, \dots, m\} \text{ such that } d_j = 0 \} \quad (\text{degr } q_j \text{ here is understood to be a rational number not exceeding } 1).$$

DEFINITION 1.2.3. Let $k \in k(A)$, $k \neq 0$. By $\Sigma_k(A)$ we shall denote the set of all real numbers α with the property that there is a $j \in \{1, \dots, m\}$ such that $d_j = 0$, $\deg q_j = k$ and

$$\alpha = \frac{1}{k} \left(\frac{\pi}{2} - \arg \mu_{j,pk} \right) \quad \text{if } k \neq 1,$$

$$\alpha = \frac{\pi}{2} - \arg (\mu_{j,p} \bmod 2\pi i) \quad \text{if } k = 1.$$

Furthermore, we define: $\Sigma_0(A) = \emptyset$ and $\Sigma(A) = \bigcup_{k \in k(A)} \Sigma_k(A)$.

The elements of $\Sigma(A)$ are comparable to the Stokes directions in the theory of homogeneous linear differential equations.

DEFINITION 1.2.4. By $\theta(A)$ we shall denote the set of all real numbers θ with the property that there is a $j \in \{1, \dots, m\}$ such that $d_j \neq 0$ and

$$d_j \theta = \operatorname{Im} \mu_{j,p} \bmod 2\pi.$$

The problem of transforming a given matrix function A into a canonical form $\overset{c}{A}$ is equivalent to that of finding a solution of the equation

$$Y(s+1) = A(s) Y(s) \overset{c}{A}(s)^{-1}$$

in some appropriate set of matrix functions Y . By $\sigma(A)$ we shall denote the matrix function corresponding to the linear mapping

$$Y \rightarrow A Y \overset{c}{A}^{-1}.$$

It is easily verified that $\sigma(\overset{c}{A})$ is a canonical form of $\sigma(A)$. Hence it follows, for example, that

$$d(\sigma(A)) = \{d - d' : d, d' \in d(A)\}.$$

It will often prove convenient to partition other matrices in the same way as a given canonical matrix $\overset{c}{A}$ (cf. (1.2.1)) associated with the particular problem under consideration. If M is an $n \times n$ matrix, the notation M_{ij} ($i, j \in \{1, \dots, m\}$) will always refer to a block of M in that partition and not to a single matrix element.

2. Existence theorems.

2.1. Preliminaries. The following sections are mainly concerned with the existence of right inverses of linear difference operators Δ_A defined on Banach spaces of holomorphic functions of the type described in the definition below.

DEFINITION 2.1.1. Let $r \in \mathbb{R}$ and let G be a closed region of the complex plane. By $B_r(G)$ we denote the Banach space of all functions f with the following properties:

- (i) f is continuous on G and holomorphic in $\operatorname{int} G$;
- (ii) $\|f\|_r = \sup_{s \in G} |s' f(s)| < \infty$.

If S is an asymptotic set of closed regions the following two statements are equivalent:

- (i) $f \in \mathcal{A}_0(S)$;
- (ii) There exists a positive number R such that $f \in B_r(S(R))$ for all $r \in \mathbb{R}$.

Let S be an asymptotic set of closed regions $S(R)$ with the additional property that $s \in S(R)$ implies $s+1 \in S(R)$ for all $R > 1$ and let $A \in \operatorname{Gl}(n; \mathcal{M}(S))$. It is easily seen that there exists a real number u such that the difference operator Δ_A maps $B_r(S(R))^n$ into $B_{r-u}(S(R))^n$ for all $r \in \mathbb{R}$ and all sufficiently large R .

DEFINITION 2.1.2. An asymptotic set of closed regions S is proper for the difference operator Δ if there exist real numbers r_0 and v , positive numbers R_0 and K , and linear mappings

$$\Lambda_{r,R}: B_r(S(R))^n \rightarrow B_{r-v}(S(R))^n$$

defined for all $r \geq r_0$ and all $R \geq R_0$ such that

- (i) $\Delta \Lambda_{r,R} f = \Lambda_{r-v,R} \Delta f = f$ for all $f \in B_r(S(R))^n$,
- (ii) $\|\Lambda_{r,R} f\|_{r-v} \leq K \|f\|_r$,
- (iii) If $r' > r \geq r_0$, then $\Lambda_{r,R}|_{B_r(S(R))^n} = \Lambda_{r',R}$.

2.2. The asymptotic sets S_θ . Let $C > 0$, $\theta \in \mathbb{R}$. We shall consider regions of the complex plane bounded by curves of the following type:

$$\sigma_C(\theta) = \{s \in \mathbb{C} : \operatorname{Re}(s \log s e^{i\theta}) = C\},$$

where $\log s$ has the principal value.

We begin by deriving some properties of these curves. First of all, note that a change from θ to $-\theta$ is equivalent to a reflection of $\sigma_C(\theta)$ with respect to the real axis. Therefore, we shall restrict the discussion to nonnegative values of θ .

If $|s| < 1$, then $\operatorname{Re} s \log |s| < 1/e$ and, consequently, $\operatorname{Re}(s \log s e^{i\theta}) < 1/e + \pi + \theta$. From now on we shall always assume that C is so large that $d(\sigma_C(\theta), 0) > 1$. We put

$$\operatorname{Im} s = x, \quad \operatorname{Re} s = \rho.$$

On the set $V = \{(x, \rho) \in \mathbb{R}^2 : x \neq 0 \text{ if } \rho \leq 0\}$ we define a function F by

$$F(x, \rho) = \rho \log \sqrt{(\rho^2 + x^2)} - x \{\arg(\rho + ix) + \theta\}.$$

By assumption, $F(x, \rho) < C$ for all (x, ρ) with the property that $\rho^2 + x^2 \leq 1$. Furthermore, we have

$$D_\rho F(x, \rho) = 1 + \log \sqrt{(\rho^2 + x^2)}.$$

Consequently, $D_\rho F(x, \rho) > 1$ whenever $\rho^2 + x^2 \geq 1$. Hence it follows that, for every $x \in \mathbb{R}$, there is a unique $\rho \in \mathbb{R}$ such that

$$(2.2.1) \quad F(x, \rho) = C.$$

Indicating this number by $\rho(x)$ and differentiating (2.2.1) with respect to x we obtain

$$(2.2.2) \quad \rho'(x) = \{\theta + \arg(\rho(x) + ix)\} \{1 + \log \sqrt{(\rho(x)^2 + x^2)}\}^{-1}, \quad x \in (-\infty, \infty).$$

Obviously, the function $\rho(x)/x \log \sqrt{(\rho(x)^2 + x^2)}$ is bounded on $|x| > 1$ and hence

$$(2.2.3) \quad \rho(x) = O\left(\frac{x}{\log x}\right), \quad |x| \rightarrow \infty.$$

For a more detailed description of the curve $\sigma_C(\theta)$ it is convenient to distinguish the following three cases.

Case 1. $0 \leq \theta < \pi/2$. The corresponding class of curves (reflected with respect to the imaginary axis) has been studied in [7]. These curves are completely contained in the right halfplane $\rho > 0$. $\rho(x)$ has an absolute minimum which is attained when $\arg(\rho(x) + ix) = -\theta$. The symmetrical case ($\theta = 0$) is represented in Fig. 1.

Case 2. $\theta = \pi/2$. In this case $\rho'(x)$ is positive for all $x \in \mathbb{R}$, but tends to 0 as $x \rightarrow -\infty$. The curve $\sigma_C(\pi/2)$ is contained in the right halfplane $\rho > 0$ and is asymptotic to the negative imaginary axis. It is easily seen that $\rho(x) = O(1/\log x)$ as $x \rightarrow -\infty$ (see Fig. 2).



FIG. 1

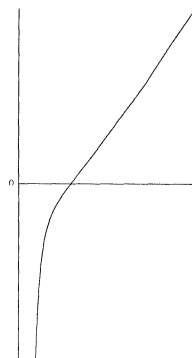


FIG. 2

Case 3. $\theta > \pi/2$. The curves in this class intersect the negative imaginary axis at $x = -C(\theta - \pi/2)^{-1}$. Furthermore, we have

$$\arg(\rho(x) + ix) + \theta > 0 \quad \text{for all } x \in \mathbb{R}$$

in this case, as can be seen from the following argument. Suppose that $\arg(\rho(x) + ix) + \theta < 0$ for some $x < 0$. This would imply that

$$\rho(x) \log \sqrt{(\rho(x)^2 + x^2)} = C + x\{\arg(\rho(x) + ix) + \theta\} > 0,$$

but this is in contradiction with the fact that $\arg(\rho(x) + ix) < -\theta < -\pi/2$ and, consequently, $\rho(x) < 0$. Hence, by (2.2.2), it follows that $\rho'(x) > 0$ for all $x \in \mathbb{R}$ (see Fig. 3).

Now let $\theta \in \mathbb{R}$ and $R > 1$. By $C(\theta, R)$ we shall denote the value of C such that $d(\sigma_C(\theta), 0) = R$.

DEFINITION 2.2.4. Let $\theta \in \mathbb{R}$. S will denote the asymptotic set of closed regions $\{S_\theta(R), R > 1\}$, where

$$S_\theta(R) = \{s \in \mathbb{C} : \operatorname{Re}(s \log s e^{i\theta}) \geq C(\theta, R)\}.$$

Furthermore, we shall put

$$S(-\pi, 0] = S_\infty, \quad S[0, \pi) = S_{-\infty}, \quad S(0, \pi] = \tilde{S}_\infty, \quad S[-\pi, 0) = \tilde{S}_{-\infty}.$$

2.3. Proper asymptotic sets. Let $\theta \in \mathbb{R}$ and $\Delta = \Delta_A$, where $A \in \operatorname{Gl}(n, \mathcal{M}(S_\theta))$. In order to prove the existence of linear mappings $\Lambda_{r,R}$ defined on the Banach spaces $B_r(S_\theta(R))^n$ and possessing the properties mentioned in Definition 2.1.2, we proceed exactly as in [7, § 12]. Thus we obtain the following addition to Proposition 4.12 in [7].

PROPOSITION 2.3.1. Let $S = \bigcup_{\theta \in [\alpha, \beta]} S_\theta$, where α and β are real numbers such that $\alpha \leq \beta$. Let $A \in \operatorname{Gl}(n; \mathcal{M}(S))$ and assume that

- (i) $[\pi/2 - \pi/k, -\pi/2] \cap \Sigma_k(A) = \emptyset$ for all $k \in k(A)$ such that $k \neq 0$;
- (ii) $[\alpha, \beta] \cap \theta(A) = \emptyset$.

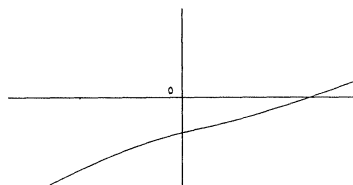


FIG. 3

Then S is proper for the difference operator Δ_A .

The next proposition is concerned with the cases $S = \bar{S}_\infty$ and $S = \bar{S}_{-\infty}$ (note that \bar{S}_∞ is the set of lower halfplanes $\{s \in \mathbb{C} : \text{Im } s \leq -R\}$, $R > 1$, whereas $\bar{S}_{-\infty}$ is a set of upper halfplanes). Although it shows some resemblance to Proposition 4.16 in [7], the assumptions made here are much more restrictive and therefore a stronger statement can be made.

PROPOSITION 2.3.2. *Let $S = \bar{S}_\infty$ or $S = \bar{S}_{-\infty}$ and $A \in \text{Gl}(n; \mathcal{M}(S))$. Assume that there exists a positive number α less than 1 and a positive number R_0 such that, for all $s \in S(R_0)$ either of the following inequalities holds:*

$$|A(s)| \leq \alpha \quad \text{or} \quad |A(s)^{-1}| \leq \alpha.$$

Then S is proper for the difference operator Δ_A .

Proof. Without loss of generality we may assume that A is continuous on $S(R_0)$ and holomorphic in $\text{int } S(R_0)$. For every $R \geq R_0$ and $r \geq 0$ we define a linear mapping $\Lambda_{r,R}$ by either of the following expressions: for all $f \in B_r(S(R))$,

(i) $\Lambda_{r,R}f(s) = f(s-1) + \sum_{h=1}^{\infty} A(s-1) \cdots A(s-h)f(s-h-1)$ if the first inequality holds, and

(ii) $\Lambda_{r,R}f(s) = -\sum_{h=0}^{\infty} A(s)^{-1} \cdots A(s+h)^{-1}f(s+h)$ if the second inequality holds. Consider the first case. For all $s \in S(R)$ we have

$$|s^r \Lambda_{r,R}f(s)| \leq \sum_{h=0}^{\infty} \alpha^h \left| \frac{s}{s-h-1} \right|^r \sup_{s \in S(R)} |s^r f(s)|$$

hence

$$\|\Lambda_{r,R}f\|_r \leq \sum_{h=0}^{\infty} \alpha^h \left(1 + \frac{h+1}{R_0} \right)^r \|f\|_r.$$

With the aid of these estimates we readily verify that the mappings $\Lambda_{r,R}$ possess the required properties. The second case can be dealt with analogously.

2.4. Analytic simplification of linear difference operators. Propositions 2.3.1 and 2.3.2 can be used to derive existence theorems for solutions of nonlinear difference equations, admitting asymptotic power series expansions in appropriate regions of the complex plane. These, in their turn, can be applied to achieve analytic simplification of homogeneous linear difference systems. Thus, for example, the statements made in Theorems 15.16 (concerning a class of nonlinear equations) and 17.13 (on block diagonalization) of [7] can now be extended immediately to all proper asymptotic sets mentioned in Proposition 2.3.1. We shall not explicitly state the generalized versions of these theorems here, but refer the reader to [7].

One immediate consequence of Proposition 2.3.1 is the following theorem.

THEOREM 2.4.1. *Let $S = \bigcup_{\theta \in [\alpha, \beta]} S_\theta$, where α and β are real numbers such that $\alpha \leq \beta$. Let $A \in \text{Gl}(n; \mathcal{M}(S))$, let $\overset{\circ}{A}$ be a canonical form of A , and let $\Phi \in \text{Gl}(n; \hat{K})$ such that $A^\Phi = \overset{\circ}{A}$. Furthermore, assume that*

- (i) $[\pi/2 - \pi/k, \pi/2] \cap \Sigma_k(\sigma(A)) = \emptyset$ for all $k \in k(\sigma(A))$ such that $k \neq 0$;
- (ii) $[\alpha, \beta] \cap \theta(\sigma(A)) = \emptyset$.

Then there exists a unique matrix function $F \in \text{Gl}(n; \mathcal{M}(S))$ such that $\hat{F} = \Phi$ and $A^F = \overset{\circ}{A}$.

With the aid of the above-mentioned extension of Theorem 17.13 in [7] the following result is obtained.

THEOREM 2.4.2. *Let $A \in \text{Gl}(n; \mathcal{M}(S(-\pi, \pi)))$ and let $\overset{\circ}{A}$ be a canonical form of A .*

(i) If $-\pi/2 \notin \Sigma(\sigma(A))$ there exists a matrix function $F \in \text{Gl}(n; \mathcal{M}(S[-\pi/2, \pi/2]))$ such that $A^F = \overset{c}{A}$.

(ii) If $\pi/2 \notin \Sigma(\sigma(A))$ there exists a matrix function $F \in \text{Gl}(n; \mathcal{M}(S(-\pi/2, \pi/2)))$ such that $A^F = \overset{c}{A}$.

Remark. The condition $A \in \text{Gl}(n; \mathcal{M}(S(-\pi, \pi)))$ may be replaced by $A \in \text{Gl}(n; \mathcal{M}(S_\theta))$ for some θ greater than $\pi/2$ in case (i) or less than $-\pi/2$ in case (ii).

The proof of Theorem 2.4.2 is roughly analogous to that of Theorem 18.13 in [7]. The difference with the latter theorem consists in the fact that here the asymptotic expansion of the matrix function F is also valid as $s \rightarrow \infty$ in the direction of the negative (case (i)) or positive (case (ii)) imaginary axis. The asymptotic behaviour of F in the opposite direction can be deduced from the following lemma (by identifying F with a vector solution of the equation $\Delta_{\sigma(A)}y = 0$).

LEMMA 2.4.3. Let $\theta \in \mathbb{R}$ and $A \in \text{Gl}(n; \mathcal{M}(S_\theta \cup S[-\pi/2, \pi/2]))$. Assume that f is a solution of the equation $\Delta_A y = 0$ with the following properties:

(i) f is continuous on the set $S_\theta(R) \cup S[-\pi/2, \pi/2](R)$ and holomorphic in its interior, for some $R > 0$.

(ii) f grows at most exponentially of order 1 as $s \rightarrow \infty$ in $S_\theta(R)$. Then f grows at most exponentially of order 1 as $s \rightarrow \infty$ in $S[-\pi/2, \pi/2](R)$ provided R is sufficiently large.

Proof. Let $s \in S[-\pi/2, \pi/2](R)$ and suppose that $|\text{Im } s| > e$ and $s \notin S_\theta(R)$. Let $n(s)$ be the smallest integer such that $s + n(s) \in S_\theta(R)$. Suppose that R is so large that A^{-1} is continuous on the set $S_\theta(R) \cup S[-\pi/2, \pi/2](R)$ and holomorphic in its interior. Then we have

$$y(s) = A(s)^{-1}A(s+1)^{-1} \cdots A(s+n(s)-1)^{-1}y(s+n(s)).$$

By assumption, there exist positive numbers c and C such that, for all $\zeta \in S_\theta(R)$,

$$|y(\zeta)| \leq C e^{c|\zeta|}.$$

Moreover, there exist positive constants d and D such that, for all $\zeta \in S_\theta(R) \cup S[-\pi/2, \pi/2](R)$,

$$|A(\zeta)^{-1}| \leq D|\zeta|^d.$$

Hence it follows that

$$(2.4.4) \quad |y(s)| \leq CD^{n(s)}|s+n(s)|^{dn(s)} e^{c|s+n(s)|}.$$

According to (2.2.3) there exists a positive constant K , independent of s , such that

$$\text{Re}(s+n(s)) \leq K \frac{|\text{Im } s|}{\log |\text{Im } s|}.$$

Since $|\text{Im } s| > e$, this implies that

$$|s+n(s)| \leq (K+1)|\text{Im } s|.$$

Hence

$$n(s) \log |s+n(s)| \leq K|\text{Im } s|\{1 + \log(K+1)\}.$$

The proof is completed by inserting the last two estimates into (2.4.4).

The next two theorems are based on Proposition 2.3.2. They can be proved by the familiar method of successive block diagonalizing transformations. The existence (and uniqueness) of these transformations can be deduced from Proposition 2.3.2 in the usual manner.

THEOREM 2.4.5. Let $S = \bar{S}_\infty$ or $\bar{S}_{-\infty}$, let $A \in \text{Gl}(n; \mathcal{M}(S))$, and let $\overset{c}{A}$ be a canonical form of A of the form (1.2.1). Assume that, for all $i, j \in \{1, \dots, m\}$ such that $i \neq j$ and $d_i = d_j$,

$$\text{deg } \text{Re}(q_i - q_j) = 1.$$

Then there exists a matrix function $F \in \text{Gl}(n; \mathcal{M}(S))$ such that

$$A^F(s) = \overset{c}{A}(s)(I_n + s^{-r}B(s))$$

where $r \in \mathbb{Q}$, $r > 1$, $B \in \text{End}(n; \mathcal{A}(S))$, and $B = \text{diag}\{B_{11}, \dots, B_{mm}\}$. Moreover, this matrix function F is determined uniquely by its asymptotic expansion.

A complete reduction of A to the canonical form in general is possible only if we drop the requirement that the asymptotic expansion of F be valid as $s \rightarrow \infty$ in both horizontal directions. The following theorem will be needed in § 4.

THEOREM 2.4.6. *Let $S \in \{S_\infty, S_{-\infty}, \tilde{S}_\infty, \tilde{S}_{-\infty}\}$. Let $A \in \text{Gl}(n; \mathcal{M}(\tilde{S}))$ and let $\overset{c}{A}$ be a canonical form of A . Assume that the conditions of Theorem 2.4.5 are satisfied. For every $\Phi \in \text{Gl}(n; \hat{K})$ such that $A^\Phi = \overset{c}{A}$, there exists a unique matrix function $F \in \text{Gl}(n; \mathcal{M}(S))$ with the following properties:*

- (i) $A^F = \overset{c}{A}$ and $\hat{F} = \Phi$;
- (ii) *There exist positive numbers k and R such that both $F(s) = O(s^k)$ and $F(s)^{-1} = O(s^k)$, $s \rightarrow \infty$, $s \in \tilde{S}(R)$.*

Remark. If A is upper or lower block triangular in the usual partition, then the same is true of F .

3. A result of Birkhoff and Trjitzinsky. Let $A \in \text{Gl}(n; \mathbb{C}\{s^{-1}\}[s])$, or, more generally, $A \in \text{Gl}(n; K)$, and let $\overset{c}{A}$ be a canonical form of A . As we mentioned in § 1.2, there exists a matrix function $\Phi \in \text{Gl}(n; \hat{K})$ with the property that

$$A^\Phi = \overset{c}{A}.$$

This section deals with the following problem.

If α is any direction in the complex plane, does there exist a matrix function F , analytic in a sector containing a half-line with direction α , such that

$$A^F = \overset{c}{A} \quad \text{and} \quad \hat{F} = \Phi?$$

So far we have been able to answer this question in the affirmative for all directions except those of the positive and negative imaginary axis. In order to include the latter directions we had to impose a rather mild condition, viz. that either $d(\sigma(A)) = \{0\}$, or else that 0 or π or both $-\pi/2$ and $\pi/2$ do not belong to $\Sigma(\sigma(A))$ (cf. Theorem 2.4.2 above and Theorem 18.18 in [7]). It is the purpose of this section to remove this last restrictive condition.

DEFINITION 3.1. A quadrant Γ is an asymptotic set of closed regions $\Gamma(R)$ ($R > 1$) of the following type:

$$\Gamma(R) = \left\{ s \in \mathbb{C} : \alpha \leq \arg(s - s_0) \leq \alpha + \frac{\pi}{2}, |s| \geq R \right\},$$

where $s_0 \in \mathbb{C}$, $\alpha = l(\pi/2)$, $l \in \mathbb{Z}$.

THEOREM 3.2. *Let $A \in \text{Gl}(n; K)$ and let $\overset{c}{A}$ be a canonical form of A . Let Γ be any quadrant. There exists a matrix function $F \in \text{Gl}(n; \mathcal{M}(\Gamma))$ such that*

$$A^F = \overset{c}{A}.$$

As a matter of fact this theorem was proved by Birkhoff and Trjitzinsky in [1]. Unfortunately, their methods are not very transparent and the argument is very hard

to follow. Here we have tried to remedy certain inaccuracies contained in this paper and have sketched a considerably simplified version of their proof.

We shall consider the case that $\alpha = -\pi/2$. All other cases can be proved analogously.

The proof consists of two steps. The first step is to carry the matrix function A into a block-triangular form by a suitable transformation. In the paper by Birkhoff and Trjitzinsky this is achieved by a rather particular method which has been exposed in earlier papers by Birkhoff alone. It is quite different from the one we used in [7], but the results are essentially the same. These can be stated as follows (cf. Lemma 9 of [1]).

PROPOSITION 3.3. *Let $\hat{S}_0 = S(-\pi, \pi)$ and $A \in \text{Gl}(n; \mathcal{M}(\hat{S}_0))$. Let \hat{A} be a canonical form of A . There exists a matrix function $F_1 \in \text{Gl}(n; \mathcal{M}(\hat{S}_0))$ such that*

$$A^{F_1} = \hat{A}(I_n + s^{-r}B(s)),$$

where $r > 1$, $B \in \text{End}(n; \mathcal{A}(\hat{S}_0))$ and $B_{ij} \equiv 0$ if $i > j$, $i, j \in \{1, \dots, m\}$.

For the proof of this proposition we refer the reader to Proposition 18.15 of [7].

The second and more delicate step is the final transformation of A^{F_1} into \hat{A} . Put $A^{F_1} = A_1$. We now search a matrix function $F \in \text{Gl}(n; \mathcal{A}(\Gamma))$ with the following properties:

- (i) $A_1^F = \hat{A}$;
- (ii) $F_{ij} \equiv 0$ if $i > j$, $i, j \in \{1, \dots, m\}$.

If $j \in \{1, \dots, m\}$ and $i \leq j$, the block F_{ij} must satisfy the following inhomogeneous difference equation:

$$(3.4) \quad Z(s+1) = (A_1)_{ii}(s)Z(s)\hat{A}_j(s)^{-1} + \sum_{i < h \leq j} (A_1)_{ih}(s)F_{hj}(s)\hat{A}_j(s)^{-1}.$$

For every $j > 1$ there are j equations that we can solve successively, beginning with F_{jj} , then $F_{j-1,j}$, etc. First, let us suppose that $d_i = d_j$ for some $i < j$. As $d_l \leq d_j$ for all $l < j$, it follows that $d_l = d_j$ for all l such that $i \leq l \leq j$.

LEMMA 3.5. *Let $i \leq j$ and assume that $d_i = d_j$. Then there exists a number $\varepsilon \in (0, \pi/2)$ and, for every l such that $i \leq l \leq j$, a matrix function $F_{lj} \in \text{Hom}(\mathcal{A}(\hat{S}_\varepsilon)^n, \mathcal{A}(\hat{S}_\varepsilon)^{n_l})$, where $\hat{S}_\varepsilon = S(-\pi + \varepsilon, \varepsilon)$, satisfying (3.4) with i replaced by l .*

The statement above can easily be deduced from Theorem 15.1 of [7] by means of induction on $j-l$. If $d_i = d_j$ for all $i, j \in \{1, \dots, m\}$ the assertion of Theorem 3.2 follows immediately. Now suppose that $d_i < d_j$ for some $j \in \{1, \dots, m\}$ and some $i < j$. Consequently, $d_l < d_j$ for all $l \leq i$. In that case the proof is completed by repeated application of Lemma 3.7 below.

DEFINITION 3.6. Let $\theta, x_0 \in \mathbb{R}$. By Γ_{θ, x_0} we shall denote the asymptotic set of closed regions $\Gamma_{\theta, x_0}(R)$ ($R > 1$) defined by

$$\Gamma_{\theta, x_0}(R) = \{s \in S_\theta(R) : \text{Im } s \leq x_0\}.$$

LEMMA 3.7. *Let $\theta, x_0 \in \mathbb{R}$, $B \in \text{End}(n; \mathcal{A}(\Gamma_{\theta, x_0}))$, and $h \in (\mathcal{A}(\Gamma_{\theta, x_0}))^n$. Let*

$$A(s) = \exp \{q(s+1) - q(s)\}(I_n + s^{-1}B(s)),$$

where $q(s) = ds \log s + \sum_{h=1}^p \mu_h s^{h/p}$, $p \in \mathbb{N}$, $d \in \mathbb{Z}/p$, $\mu_h \in \mathbb{C}$ for all $h \in \{1, \dots, p\}$. Suppose that $d < 0$ and $d\theta \neq \text{Im } \mu_p \pmod{2\pi}$. Then the equation

$$(3.8) \quad \Delta_A y = h$$

possesses a solution $y \in \bigcap_{x < x_0} \mathcal{A}(\Gamma_{\theta, x})^n$.

Proof. The homogeneous equation $\Delta_A y = 0$ possesses a fundamental matrix Y of the form

$$Y(s) = Z(s)s^G e^{q(s)},$$

where G is a constant matrix and $Z \in \text{Gl}(n; \mathcal{A}(\Gamma_{\theta, x_0}))$. Since, by assumption, $\theta \neq 1/d$ ($\text{Im } \mu_p \bmod 2\pi$) and as μ_p obviously is determined modulo $2\pi i$ by A , we may assume that

$$d\theta < \text{Im } \mu_p < d\theta + 2\pi.$$

Let R be a positive number such that Z is holomorphic in $\text{int } \Gamma_{\theta, x_0}(R)$ and represented asymptotically by \hat{Z} as $s \rightarrow \infty$ in $\Gamma_{\theta, x_0}(R)$. Let s_0 denote the point on the boundary of $S_\theta(R)$ with the property that $\text{Im } s_0 = x_0$. Consider the linear mapping Λ of $B_0(\Gamma_{\theta, x_0}(R))$ (cf. Definition 2.1.1) defined by the following formula:

$$\Lambda f(s) = Y(s) \int_{C(s)} d\zeta \frac{Y(\zeta+1)^{-1} f(\zeta)}{1 - \exp \{2\pi i(s - \zeta)\}} + A(s)^{-1} f(s),$$

where $f \in B_0(\Gamma_{\theta, x_0}(R))$, $s \in \Gamma_{\theta, x_0}(R)$ such that $\text{Im } s < x_0$ and $C(s)$ is a path going from s_0 to infinity in such a way that it intersects the line $\text{Im } \zeta = \text{Im } s$ exactly once, in a point between s and $s+1$.

It can easily be verified that the vector function Λh is a solution of the equation (3.8). In order to prove that it has the desired asymptotic properties, we shall show that, for all $r > 0$ and all $x < x_0$, Λ maps the Banach space $B_r(\Gamma_{\theta, x_0}(R))$ into $B_{r+v}(\Gamma_{\theta, x}(R))$, where v is some fixed real number.

For all $s \in \Gamma_{\theta, x_0}(R)$ let s' denote the point on the boundary of Γ_{θ, x_0} with the property that

$$\text{Re}(s' \log s' e^{i\theta}) = \text{Re} \left\{ \left(s + \frac{1}{2}\right) \log \left(s + \frac{1}{2}\right) e^{i\theta} \right\}.$$

Let $C_1(s)$ be the path from ∞ to s' such that

$$\text{Re}(\zeta \log \zeta e^{i\theta}) = \text{Re}(s' \log s' e^{i\theta}), \quad \zeta \in C_1(s)$$

and let $C_2(s)$ denote the directed line segment from s' to s_0 (see Fig. 4). Let $r > 0$, $x < x_0$, and $f \in B_r(\Gamma_{\theta, x}(R))$. Putting

$$Y(s) \int_{C_i(s)} d\zeta \frac{Y(\zeta+1)^{-1} f(\zeta)}{1 - \exp \{2\pi i(s - \zeta)\}} = I_i(s), \quad i = 1, 2$$

we have

$$(3.9) \quad \Lambda f(s) = I_1(s) + I_2(s) + A(s)^{-1} f(s).$$

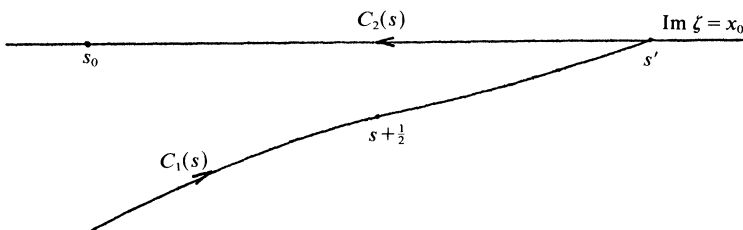


FIG. 4

The second term may be written as follows:

$$(3.10) \quad I_2(s) = Y(s) Y(s')^{-1} \exp \{2\pi i(s' - s)\} \exp \{2\pi i(s - s')\} I_2(s').$$

Noting that, for all $s \in \Gamma_{\theta, x}(R)$ and all $\zeta \in C_2(s)$,

$$\left| \frac{\exp \{2\pi i(s - s')\}}{1 - \exp \{2\pi i(s - \zeta)\}} \right| = |\exp \{-2\pi i(s - \zeta)\} - 1|^{-1} \leq [1 - \exp \{2\pi(x - x_0)\}]^{-1}$$

and taking into account the rapid decrease of $Y(\zeta + 1)^{-1}$ along $C_2(s)$ we readily verify that

$$\sup_{s \in \Gamma_{\theta, x}(R)} |(s')^{r+d} \exp \{2\pi i(s - s')\} I_2(s')| < \infty.$$

The term $I_1(s)$ in (3.9) and the product

$$Y(s) Y(s')^{-1} \exp \{2\pi i(s' - s)\}$$

in (3.10) can be dealt with by the methods used in [7, § 12]. Indeed, the integral $I_1(s)$ is similar to the one figuring in (12.2) of [7], where the above product can be estimated in much the same way as the integrand of $I_+(s)$ defined on p. 71 of [7]. Thus we find that $\Lambda f \in B_{r+d}(\Gamma_{\theta, x}(R))$, provided R is sufficiently large. Hence it follows that $\Lambda h \in \mathcal{A}(\Gamma_{\theta, x})^n$ for all $x < x_0$. This concludes the proof of the lemma.

Remark. With the aid of Lemma 3.7 we can prove a slightly stronger statement than the one made in Theorem 3.2, namely, the existence, for all $x \in \mathbb{R}$ and all $\theta \notin \theta(\sigma(A))$, of a matrix function $F \in \text{Gl}(n; \mathcal{M}(\Gamma_{\theta, x}))$ with the property that $A^F = \overset{\circ}{A}$.

4. The Stokes phenomenon.

4.1. A preliminary transformation. In the remaining sections we shall determine the “maximal asymptotic sets” for and study the connection between different fundamental matrix solutions of the linear homogeneous difference equation

$$(4.1.1) \quad \Delta_A y = 0$$

where $A \in \text{Gl}(n; \mathbb{C}\{s^{-1}\}[s])$.

In order to avoid the complications caused by the intermingling of different types of Stokes phenomena (associated with different levels) we shall make the simplifying assumption that the set $d(A)$ defined in § 1.2 has m distinct elements.

Let $\overset{\circ}{A}$ be a canonical form of A and let U denote the asymptotic set of closed regions $U(R)$ ($R > 1$), defined by (0.2). According to Theorem 18.16 in [7] there exists a matrix function $T \in \text{Gl}(n; \mathcal{M}(U))$ such that

$$A^T(s) = \overset{\circ}{A}(s)(I_n + s^{-r}B(s)),$$

where $r > 1$, $B \in \text{End}(n; \mathcal{A}(U))$ and $B_{ij} \equiv 0$ if $i > j$, $i, j \in \{1, \dots, m\}$. (Actually, Theorem 18.16 only states the existence of matrix functions $T_1 \in \text{Gl}(n; \mathcal{M}(S[-\pi, \pi]))$ and $T_2 \in \text{Gl}(n; \mathcal{M}(S(-\pi, \pi]))$ with analogous properties, but these can be seen to coincide in some sector, provided $\hat{T}_1 = \hat{T}_2$.) Let S be an asymptotic set of closed regions with the property that $S(R) \subset U(R)$ for all $R > 1$. In the following sections we shall consider fundamental matrix solutions of (4.1.1) of the form

$$Y = T F \overset{\circ}{Y},$$

where F is a solution of the equation

$$(4.1.2) \quad Z(s+1) = A^T(s) Z(s) \overset{\circ}{A}(s)^{-1}$$

with the properties that $F \in \text{Gl}(n; \mathcal{A}(S))$, $F(\infty) = I_n$, and $F_{ij} \equiv 0$ if $i > j$, $i, j \in \{1, \dots, m\}$.

4.2. Connection matrices. Let G_1 and G_2 be open regions in $U(1)$ such that $G_1 \cap G_2 \neq \emptyset$. Let F_1 and F_2 be solutions of (4.1.2), holomorphic in G_1 and G_2 , respectively.

DEFINITION 4.2.1. The connection matrix of the pair (F_1, F_2) is the matrix function P defined by the expression

$$P = \overset{c}{Y}^{-1} F_1^{-1} F_2 \overset{c}{Y}.$$

Obviously, P is a periodic matrix function of period 1. If both F_1 and F_2 are upper block-triangular in the usual partition, then so is P .

In what follows we shall always be concerned with the case that one of the two regions G_1 and G_2 is a lower or an upper halfplane. It can easily be verified that (4.1.2) possesses a unique formal solution $\sum_{h=0}^{\infty} F_h s^{-h/p}$ ($p \in \mathbb{N}$), with the property that $F_0 = I_n$. Hence, according to Theorem 2.4.6, there exist four unique matrix functions

$$\begin{aligned} F^{\infty} &\in \text{Gl}(n; \mathcal{A}(S_{\infty})), & F^{-\infty} &\in \text{Gl}(n; \mathcal{A}(S_{-\infty})), \\ \tilde{F}^{\infty} &\in \text{Gl}(n; \mathcal{A}(\tilde{S}_{\infty})), & \tilde{F}^{-\infty} &\in \text{Gl}(n; \mathcal{A}(\tilde{S}_{-\infty})), \end{aligned}$$

with the properties mentioned in the theorem. Moreover, all four matrix functions are upper block triangular (cf. the remark below Theorem 2.4.6). Let P^{∞} and $P^{-\infty}$ denote the connection matrices of the pairs $(F^{\infty}, \tilde{F}^{-\infty})$ and $(F^{-\infty}, \tilde{F}^{\infty})$, respectively. Obviously, these too are upper block triangular. Let \hat{A} have the form (1.2.1). For all $i, j \in \{1, \dots, m\}$ we have

$$P_{ij}^{\infty}(s) = \exp(q_j(s) - q_i(s)) s^{-G_i} (F^{\infty}(s)^{-1} \tilde{F}^{-\infty}(s))_{ij} s^{G_j}.$$

If $i < j$, due to the fact that $d_i < d_j$, the first factor on the right-hand side of this identity decreases very rapidly as $\text{Re } s \rightarrow -\infty$. In view of the growth properties of F^{∞} and $\tilde{F}^{-\infty}$ this implies that P_{ij}^{∞} tends to zero as $\text{Re } s \rightarrow -\infty$ and hence must vanish identically.

Similarly, it can be seen that $P_{ij}^{-\infty} \equiv 0$ if $i \neq j$. Now consider the diagonal blocks F_{ii}^{∞} and $\tilde{F}_{ii}^{-\infty}$, $i \in \{1, \dots, m\}$. Both satisfy the equation

$$\begin{aligned} Z(s+1) &= A_{ii}^T(s) Z(s) \hat{A}_i(s)^{-1} \\ &= \left(1 + \frac{1}{s}\right)^{N_i} (I_{n_i} + s^{-r} B_{ii}(s)) Z(s) \left(1 + \frac{1}{s}\right)^{-N_i}. \end{aligned}$$

This equation has a solution $F_i \in \text{Gl}(n_i; \mathcal{A}(S(-\pi, \pi)))$ represented by the infinite product

$$(4.2.2) \quad F_i(s) = s^{N_i} \prod_{n=0}^{\infty} (I_{n_i} + B_i(s+n))^{-1} s^{-N_i}, \quad s \in U(R_0)$$

and a solution $\tilde{F}_i \in \text{Gl}(n_i; \mathcal{A}(\tilde{S}_{\infty}) \cap \mathcal{A}(\tilde{S}_{-\infty}))$ represented by

$$\tilde{F}_i(s) = s^{N_i} \prod_{n=1}^{\infty} (I_{n_i} + B_i(s-n)) s^{-N_i}, \quad s \in \tilde{S}_{\infty}(R_0) \cup \tilde{S}_{-\infty}(R_0),$$

where $B_i(s) = s^{-r} s^{-N_i} B_{ii}(s) s^{N_i}$ and R_0 is some sufficiently large positive number. It is easily seen that

$$\lim_{\text{Res} \rightarrow \infty} F_i(s) = \lim_{\text{Res} \rightarrow -\infty} \tilde{F}_i(s) = I_{n_i}.$$

Furthermore, if R_0 is sufficiently large, F_i and \tilde{F}_i as well as their inverses are bounded on $\tilde{S}_\infty(R_0) \cup \tilde{S}_{-\infty}(R_0)$. Hence, by Theorem 2.4.6, F_i must coincide with F_{ii}^∞ in a lower halfplane and with $F_{ii}^{-\infty}$ in an upper halfplane, whereas \tilde{F}_i must coincide with $\tilde{F}_{ii}^{-\infty}$ in a lower halfplane and with \tilde{F}_{ii}^∞ in an upper halfplane. Consequently, the connection matrices P_{ii}^∞ and $P_{ii}^{-\infty}$ can both be represented by the infinite product

$$\prod_{n=-\infty}^{\infty} (I_{n_i} + B_i(s-n)), \quad i \in \{1, \dots, m\}$$

in a lower and an upper halfplane, respectively. We shall assume that R_0 is so large that F^∞ and $(F^\infty)^{-1}$ are bounded on the (closed) halfplane $\tilde{S}_\infty(R_0)$, and, moreover, $F^{-\infty}$ and its inverse have the same properties with respect to $\tilde{S}_{-\infty}(R_0)$.

Now let G be an open region in $U(R_0)$ such that

$$G = \bigcup_{x \in \mathbb{R}^+} G + x,$$

and $\sup_{s \in G} \operatorname{Im} s = -\inf_{s \in G} \operatorname{Im} s = \infty$.

Let F be a solution of (4.1.2), holomorphic in G and with diagonal blocks $F_{ii} = F_i$, where F_i is defined by (4.2.2), for all $i \in \{1, \dots, m\}$. F may be continued analytically to a holomorphic function in

$$\bigcup_{x \in \mathbb{R}} G + x \cap U(R_0).$$

This function will again be denoted by F .

In the following lemma we consider the connection matrices of (F^∞, F) and $(F^{-\infty}, F)$.

LEMMA 4.2.3. *Let $Y_i = F_i \overset{c}{Y}_i$, $i \in \{1, \dots, m\}$ and let P denote the matrix function defined by*

$$P(s) = \begin{cases} \overset{c}{Y}(s)^{-1} F^\infty(s)^{-1} F(s) \overset{c}{Y}(s) & \text{if } s \in \tilde{S}_\infty(R_0), \\ \overset{c}{Y}(s)^{-1} F^{-\infty}(s)^{-1} F(s) \overset{c}{Y}(s) & \text{if } s \in \tilde{S}_{-\infty}(R_0). \end{cases}$$

For all $i, j \in \{1, \dots, m\}$ and all $s \in \tilde{S}_\infty(R_0) \cup \tilde{S}_{-\infty}(R_0)$ the following identity holds:

$$P_{ij}(s) = \lim_{n \rightarrow \infty} Y_i(s-n)^{-1} F_{ij}(s-n) \overset{c}{Y}_j(s-n).$$

Proof. We shall prove the statement for all $s \in \tilde{S}_\infty(R_0)$, by means of induction on $j-i$. If $j-i=1$ we have, for all $s \in \tilde{S}_\infty(R_0)$ and all $n \in \mathbb{Z}$,

$$P_{ij}(s) = Y_i(s-n)^{-1} F_{ij}(s-n) \overset{c}{Y}_j(s-n) + \overset{c}{Y}_i(s-n)^{-1} (F^\infty)^{-1}_{ij}(s-n) Y_j(s-n).$$

Due to the fact that $d_i < d_j$ while $(F^\infty)^{-1}$ is bounded on $\tilde{S}_\infty(R_0)$, the second term on the right-hand side of this identity tends to zero as $n \rightarrow \infty$.

Now suppose that $j-i > 1$ and that the statement is true for all pairs of indices (k, l) such that $l-k < j-i$. Then we have

$$\begin{aligned} P_{ij}(s) &= Y_i(s-n)^{-1} F_{ij}(s-n) \overset{c}{Y}_j(s-n) \\ &= \sum_{i < h \leq j} \overset{c}{Y}_i(s-n)^{-1} (F^\infty)^{-1}_{ih}(s-n) Y_h(s-n) Y_h(s-n)^{-1} F_{hj}(s-n) \overset{c}{Y}_j(s-n). \end{aligned}$$

Again the product $\overset{c}{Y}_i(s-n)^{-1} (F^\infty)^{-1}_{ih}(s-n)^{-1} Y_h(s-n)$ tends to zero as $n \rightarrow \infty$ for all $h > i$. By assumption, for all $h < j$ the product $Y_h(s-n)^{-1} F_{hj}(s-n) \overset{c}{Y}_j(s-n)$ tends to a finite limit, namely, $P_{hj}(s)$, as $n \rightarrow \infty$. Consequently, the right-hand side of the above

identity tends to zero as $n \rightarrow \infty$ and the result follows. For $s \in \bar{S}_{-\infty}(R_0)$ the proof is analogous.

4.3. Maximal asymptotic sets. Let $\theta_0 = \max \{ \theta \in \theta(\sigma(A)) : \theta \leq 0 \}$ and let all elements of $\theta(\sigma(A))$ be numbered in such a way that $\theta_N < \theta_{N+1}$, $N \in \mathbb{Z}$. For all $N \in \mathbb{Z}$ and all $i, j \in \{1, \dots, m\}$, $n_{ij}(N)$ will denote the smallest integer not less than

$$\frac{1}{2\pi} \{ (d_i - d_j) \theta_N - \text{Im} (\mu_{i,p} - \mu_{j,p}) \}.$$

By the definition of $\theta(\sigma(A))$, there exists an integer M such that

$$n_{ij}(N) = \frac{1}{2\pi} \{ (d_i - d_j) \theta_M - \text{Im} (\mu_{i,p} - \mu_{j,p}) \}$$

and an integer M' such that

$$n_{ij}(N) + 1 = \frac{1}{2\pi} \{ (d_i - d_j) \theta_{M'} - \text{Im} (\mu_{i,p} - \mu_{j,p}) \}.$$

Suppose that $i < j$. This implies that $d_i < d_j$. Then, obviously $M' < M \leq N$. Hence it follows that

$$(4.3.1) \quad 0 \leq n_{ij}(N-1) - n_{ij}(N) \leq 1, \quad i, j \in \{1, \dots, m\}, \quad i < j, \quad N \in \mathbb{Z}.$$

For all $N \in \mathbb{Z}$ we define an upper block-triangular matrix function F_N by means of the following recursive relation for the blocks $(F_N)_{ij}$:

$$(4.3.2) \quad \begin{aligned} (F_N)_{ii} &= F_i \quad (\text{defined in (4.2.2)}), \\ (F_N)_{ij}(s) &= Y_i(s) \int_{C(s)} d\zeta \frac{\exp \{ 2n_{ij}(N) \pi i (s - \zeta) \}}{1 - \exp \{ 2\pi i (s - \zeta) \}} I_{ij}^N(\zeta) \overset{c}{Y}_j(s)^{-1}, \quad i < j, \end{aligned}$$

$s \in \text{int } U(R_0)$, where

$$(4.3.3) \quad I_{ij}^N(\zeta) = Y_i(\zeta + 1)^{-1} \sum_{i < h \leq j} A_{ih}^T(\zeta) (F_N)_{hj}(\zeta) \overset{c}{Y}_j(\zeta)$$

and $C(s)$ is a contour in $U(R_0)$, enclosing the negative imaginary axis as well as the points $s - n$, $n \in \mathbb{N}$, but not s (see Fig. 5).

LEMMA 4.3.4. *Let R and R' be positive numbers such that $R' > R > R_0$. Let $U_0 = U(R) \setminus U(R')$. There exist positive constants c and C such that, for all $i, j \in \{1, \dots, m\}$ and all $N \in \mathbb{Z}$,*

$$\sup_{s \in U_0} |Y_i(s)^{-1} (F_N)_{ij}(s) \overset{c}{Y}_j(s)| \leq C e^{c|N|}.$$

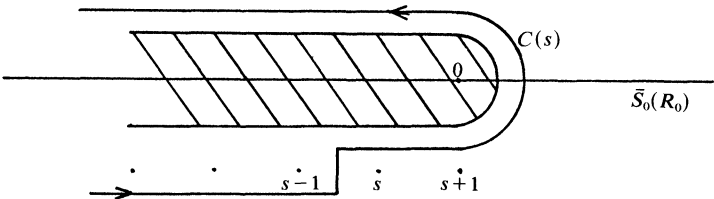


FIG. 5

Proof. For all $k \in \{1, \dots, m\}$ we define $U_k = U(R - k\varepsilon) \setminus U(R' + k\varepsilon)$, where ε is a sufficiently small positive number such that $U_m \subset U(R_0)$. For all $(s, \zeta) \in U_m \times U_m$ we have

$$|\exp \{2n_{ij}(N)\pi i(s - \zeta)\}| \leq \exp \{4(R' + m\varepsilon)\pi |n_{ij}(N)|\}.$$

Hence, in view of (4.3.1), we obtain the inequality

$$(4.3.5) \quad |\exp \{2n_{ij}(N)\pi i(s - \zeta)\}| \leq C_0 e^{c_0|N|},$$

where c_0 and C_0 are positive numbers independent of N .

If $s \in U_k$ for some $k \in \{1, \dots, m-1\}$ the contour $C(s)$ in (4.3.2) may be chosen in such a way that $C(s) \subset U_{k+1}$ and $d(\zeta, s + \mathbb{Z}) \geq \varepsilon$ for all $\zeta \in C(s)$. Then there exists a positive number K such that

$$(4.3.6) \quad |1 - \exp \{2\pi i(s - \zeta)\}|^{-1} \leq K \quad \text{for all } s \in U_{m-1} \text{ and all } \zeta \in C(s).$$

We shall prove the lemma by means of induction on $j - i$. If $j - i = 1$ the expression in (4.3.3) is reduced to

$$I_{ij}^N(\zeta) = Y_i(\zeta + 1)^{-1} A_{ij}^T(\zeta) Y_j(\zeta).$$

Due to the fact that $d_i < d_j$, this function tends to zero very rapidly as $\text{Re } \zeta \rightarrow -\infty$, uniformly on U_m . Consequently, the integrals

$$\int_{C(s)} |I_{ij}^N(\zeta)| |d\zeta|, \quad s \in U_{m-1}$$

exist and are bounded by a constant, independent of N . With (4.3.5) and (4.3.6) it follows that, for all $N \in \mathbb{Z}$,

$$\sup_{s \in U_{m-1}} |Y_i(s)^{-1} (F_N)_{ij}(s) \overset{c}{Y}_j(s)| \leq C_1 e^{c_1|N|},$$

where c_1 and C_1 are positive constants independent of N .

Now let $j - i = k > 1$ and suppose that for all $l < k$ there exist positive numbers c_l and C_l such that

$$(4.3.7) \quad \sup_{s \in U_{m-l}} |Y_g(s)^{-1} (F_N)_{gh}(s) \overset{c}{Y}_h(s)| \leq C_l e^{c_l|N|}$$

for all $N \in \mathbb{Z}$ and all $h, g \in \{1, \dots, m\}$ such that $h - g = l$. Then we have, for all $N \in \mathbb{Z}$, all $h \in \{1, \dots, m\}$ such that $i < h < j$ and all $s \in U_{m-j+h}$,

$$|Y_i(s+1)^{-1} A_{ih}^T(s) (F_N)_{hj}(s) \overset{c}{Y}_j(s)| \leq |Y_i(s+1)^{-1} A_{ih}^T(s) Y_h(s)| C_{j-h} e^{c_{j-h}|N|}.$$

Inserting this into (4.3.3) and using the estimates (4.3.5) and (4.3.6) we conclude that there exist positive numbers c_k and C_k such that (4.3.7) holds for $l = k$ as well. Hence it follows that for all $l \in \{1, \dots, m-1\}$ there exist positive numbers c_l and C_l such that (4.3.7) holds. Since $U_0 \subset U_l$ for all $l \in \{1, \dots, m-1\}$ this proves the lemma.

With the aid of residue calculus it is readily verified that, for all $i, j \in \{1, \dots, m\}$ such that $i < j$ and all $N \in \mathbb{Z}$, the matrix function $(F_N)_{ij}$ satisfies the equation

$$Z(s+1) = A_{ii}^T(s) Z(s) \overset{c}{A}_j(s)^{-1} + \sum_{i < h \leq j} A_{ih}^T(s) (F_N)_{hj}(s) \overset{c}{A}_j(s)^{-1}.$$

Hence it follows that, for all $N \in \mathbb{Z}$, F_N is a solution of (4.1.2).

THEOREM 4.3.8. For all $N \in \mathbb{Z}$ let P_N^+ and P_N^- denote the connection matrices of (F^∞, F_N) and $(F^{-\infty}, F_N)$, respectively. Let $R_1 > R_0$. There exist positive numbers d and D such that, for all $j \in \{1, \dots, m\}$, all $i < j$ and all $N \in \mathbb{Z}$, the following inequalities hold:

$$\sup_{s \in \bar{S}_\infty(R_1)} |(P_N^+)_{ij}(s) \exp \{-2(n_{ij}(N) - 1)\pi i s\}| \leq D e^{d|N|},$$

$$\sup_{s \in \bar{S}_{-\infty}(R_1)} |(P_N^-)_{ij}(s) \exp \{-2n_{ij}(N)\pi i s\}| \leq D e^{d|N|}.$$

COROLLARY 4.3.9. There exists a positive number R such that

$$\lim_{N \rightarrow \infty} F_N(s) = F^\infty(s) \quad \text{if } \operatorname{Im} s \leq -R,$$

$$\lim_{N \rightarrow -\infty} F_N(s) = F^{-\infty}(s) \quad \text{if } \operatorname{Im} s \geq R.$$

Proof. For all $N \in \mathbb{Z}$, all $i, j \in \{1, \dots, m\}$ such that $i < j$ and all $s \in \bar{S}_\infty(R_0)$ we have

$$(4.3.10) \quad (F_N)_{ij}(s) = F_{ij}^\infty(s) + \sum_{i \leq h < j} F_{ih}^\infty(s) \overset{c}{Y}_h(s) (P_N^+)_{hj}(s) \overset{c}{Y}_j(s)^{-1}.$$

Applying Theorem 4.3.8 and noting that, according to (4.3.1),

$$n_{hj}(0) - n_{hj}(N) \leq N \quad \text{for all } h < j, \quad N \in \mathbb{Z},$$

we obtain the following inequality for all $s \in \bar{S}_\infty(R_1)$:

$$|(F_N)_{ij}(s) - F_{ij}^\infty(s)| \leq \sum_{i \leq h < j} |F_{ih}^\infty(s) \overset{c}{Y}_h(s) \overset{c}{Y}_j(s)^{-1}| D_{hj}(s) \exp \{N(d - 2\pi \operatorname{Im} s)\},$$

where $D_{hj}(s) = D \exp \{2\pi(n_{hj}(0) + 1) \operatorname{Im} s\}$. If we take $R > \max(R_1, d/2\pi)$ the first statement follows immediately. The second statement is proved analogously.

Proof of Theorem 4.3.8. Let C_0 be a U-shaped contour in the interior of $U(R_1) \setminus U(R_0)$ enclosing the negative imaginary axis. By means of residue calculus it is easily shown that

$$Y_i(s)^{-1} (F_N)_{ij} \overset{c}{Y}_j(s)^{-1} = (P_N)_{ij}(s) + \sum_{l=1}^{\infty} I_{ij}^N(s-l),$$

where I_{ij}^N is defined by (4.3.3) and

$$(P_N)_{ij}(s) = \int_{C_0} d\zeta \frac{\exp \{2n_{ij}(N)\pi i(s-\zeta)\}}{1 - \exp \{2\pi i(s-\zeta)\}} I_{ij}^N(\zeta),$$

for all $N \in \mathbb{Z}$, all $i, j \in \{1, \dots, m\}$ such that $i < j$ and all s with the property that $|\operatorname{Im} s| \geq R_1$.

Consequently, the following identity holds for all $n \in \mathbb{N}$:

$$Y_i(s-n)^{-1} (F_N)_{ij}(s-n) \overset{c}{Y}_j(s-n) = (P_N)_{ij}(s) + \sum_{l=n}^{\infty} I_{ij}^N(s-l).$$

Making $n \rightarrow \infty$ and applying Lemma 4.2.3 we find

$$(P_N)_{ij}(s) = \begin{cases} (P_N^+)_{ij}(s) & \text{if } \operatorname{Im} s \leq -R_1, \\ (P_N^-)_{ij}(s) & \text{if } \operatorname{Im} s \geq R_1. \end{cases}$$

Hence it follows that

$$(4.3.11) \quad \exp \{-2(n_{ij}(N) - 1)\pi i s\} (P_N^+)_{ij}(s) = \int_{C_0} d\zeta \frac{\exp \{-2(n_{ij}(N) - 1)\pi i \zeta\}}{\exp \{-2\pi i(s-\zeta)\} - 1} I_{ij}^N(\zeta)$$

for all $s \in \bar{S}_\infty(R_1)$, whereas

$$(4.3.12) \quad \exp \{-2n_{ij}(N)\pi i s\} (P_N^-)_{ij}(s) = \int_{C_0} d\zeta \frac{\exp \{-2n_{ij}(N)\pi i \zeta\}}{1 - \exp \{2\pi i(s - \zeta)\}} I_{ij}^N(\zeta)$$

for all $s \in \bar{S}_{-\infty}(R_1)$. Writing

$$I_{ij}^N(\zeta) = \sum_{i < h \leq j} Y_i(\zeta + 1)^{-1} A_{ih}^T(\zeta) Y_h(\zeta) Y_h(\zeta)^{-1} (F_N)_{hj}(\zeta) \overset{c}{Y}_j(\zeta)$$

and using Lemma 4.3.4 we obtain, for all $\zeta \in C_0$, the estimate

$$I_{ij}^N(\zeta) \leq C e^{c|N|} \sum_{i < h \leq j} |Y_i(\zeta + 1)^{-1} A_{ih}^T(\zeta) Y_h(\zeta)|,$$

where c and C are positive constants. As $d_i < d_h$ if $i < h$, the integral

$$\int_{C_0} |d\zeta| |Y_i(\zeta + 1)^{-1} A_{ih}^T(\zeta) Y_h(\zeta)|$$

is convergent for all $i < h$. Furthermore, the functions

$$\varphi(s, \zeta) = [\exp \{-2\pi i(s - \zeta)\} - 1]^{-1}$$

and

$$\psi(s, \zeta) = [1 - \exp \{2\pi i(s - \zeta)\}]^{-1}$$

are obviously bounded on $\bar{S}_\infty(R_1) \times C_0$ and $\bar{S}_{-\infty}(R_1) \times C_0$, respectively. With (4.3.1) it is now easily verified that the expressions in (4.3.11) and (4.3.12) lead to estimates of the required form.

Remark. The matrix elements of P_N^{-1} (i.e., the connection matrix of (F_N, F^∞) and $(F_N, F^{-\infty})$) can be computed from F_N by means of the following recursive relation:

$$\begin{aligned} (P_N^{-1})_{ij} &= \delta_{ij} \quad \text{if } i \geq j, \\ &= \delta_{ij} - \sum_{i < l \leq j} \int_{C_0} d\zeta \frac{\exp \{2n_{il}(N)\pi i(s - \zeta)\}}{1 - \exp \{2\pi i(s - \zeta)\}} I_{il}^N(\zeta) (P_N^{-1})_{lj} \quad \text{if } i < j \end{aligned}$$

where I_{il}^N is defined by (4.3.3).

The next, and final, proposition is concerned with the asymptotic behaviour of the matrix functions F_N .

PROPOSITION 4.3.13. *Let $N \in \mathbb{Z}$ and $S_N^* = \bigcup_{\theta_N < \theta < \theta_{N+1}} S_\theta$. We have*

$$F_N \in \text{Gl}(n; \mathcal{A}(S_N^*)).$$

Moreover, F_N is the unique solution of (4.1.2) which is analytic in a right half plane and possesses the properties mentioned in Theorem 4.3.8.

Proof. Let $\theta \in (\theta_N, \theta_{N+1})$ and let

$$S_\theta^- = S_\theta \cap \bar{S}_\infty.$$

Putting

$$d_h - d_j = d_{hj}, \quad \mu_h - \mu_j + 2(n_{hj}(N) - 1)\pi i = \mu_{hj}^N$$

for all $h, j \in \{1, \dots, m\}$ and all $N \in \mathbb{Z}$, and using Theorem 4.3.8 we find

$$|\overset{c}{Y}_h(s) (P_N^+)_{hj}(s) \overset{c}{Y}_j(s)^{-1}| = \exp \{(d_{hj}s \log s + \mu_{hj}^N s)(1 + o(1))\}, \quad s \rightarrow \infty, \quad s \in S_\theta^-(R_1),$$

for all $j \in \{1, \dots, m\}$ and all $h < j$. By Definition 2.2.4,

$$\text{Re } s \log s e^{i\theta} \geq C(\theta, R_1)$$

for all $s \in S_\theta(R_1)$, hence

$$\operatorname{Re}(d_{hj}s \log s + \mu_{hj}^N s) \leq d_{hj}C(\theta, R_1) + \operatorname{Re}\{(\mu_{hj}^N - id_{hj}\theta)s\}$$

if $h < j$ and $s \in S_\theta(R_1)$. Now, $\theta \in (\theta_N, \theta_{N+1})$ implies that

$$0 < d_{hj}\theta - \operatorname{Im} \mu_{hj}^N < 2\pi.$$

With the aid of the above inequalities we readily verify that

$$\overset{c}{Y}_h(s)(P_N^+)_{hj}(s)\overset{c}{Y}_j(s)^{-1} \sim 0, \quad s \rightarrow \infty, \quad s \in S_\theta^-(R),$$

if $h < j$ and R is a sufficiently large number. With (4.3.10) we conclude that

$$F_N - F^\infty \in \operatorname{Gl}(n; \mathcal{A}_0(S_\theta^-)).$$

In a similar manner we prove that

$$F_N - F^{-\infty} \in \operatorname{Gl}(n; \mathcal{A}_0(S_\theta \cap \bar{S}_{-\infty})).$$

As F^∞ and $F^{-\infty}$ are represented asymptotically by the series $\sum_{h=0}^\infty F_h s^{-h/p}$ as $s \rightarrow \infty$ in $\bar{S}_\infty(R_1)$ and $\bar{S}_{-\infty}(R_1)$, respectively, it remains to be shown that F_N admits the same asymptotic expansion as $s \rightarrow \infty$ in a “strip” of the form $S_\theta(R) \cap \{s \in \mathbb{C} : |\operatorname{Im} s| \leq R\}$, where R is a suitable positive number. This follows immediately from a Phragmén–Lindelöf-type of argument (cf. [15, p. 180]). Thus we conclude that $F_N \in \operatorname{Gl}(n; \mathcal{A}(S_\theta))$ for all $\theta \in (\theta_N, \theta_{N+1})$ and, consequently, $F_N \in \operatorname{Gl}(n; \mathcal{A}(S_N^*))$. Since here we have not used any properties of F_N , except for those mentioned in Theorem 4.3.8, the second statement now follows from Theorem 2.4.1.

In conclusion we can say that, unless $P_{N-1} = P_N$ or $P_N = P_{N+1}$, the asymptotic sets S_N^* mentioned in Proposition 4.3.13 are maximal in the following sense: for every $N \in \mathbb{Z}$ there exists a (unique) matrix function $F_N \in \operatorname{Gl}(n; \mathcal{A}(S_N^*))$ with the property that $A^{F_N} = \overset{c}{A}$, whereas $F_N \notin \operatorname{Gl}(n; \mathcal{A}(S_{\theta_N}))$ and $F_N \notin \operatorname{Gl}(n; \mathcal{A}(S_{\theta_{N+1}}))$. Analogously, there exist maximal asymptotic sets \tilde{S}_N , defined by

$$\tilde{S}_N = \bigcup_{\theta_N < \theta < \theta_{N+1}} e^{i\pi} S_{\theta+\pi}, \quad N \in \mathbb{Z}$$

and unique matrix functions $\tilde{F}_N \in \operatorname{Gl}(n; \mathcal{A}(\tilde{S}_N))$ such that $A^{\tilde{F}_N} = \overset{c}{A}$. The Stokes phenomenon in the class of difference equations considered in this section can be completely described by determining the connection matrices of (F_N, F_{N+1}) and $(\tilde{F}_N, \tilde{F}_{N+1})$, $N \in \mathbb{Z}$, and the connection matrices P^∞ and $P^{-\infty}$ defined below Definition 4.2.1.

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